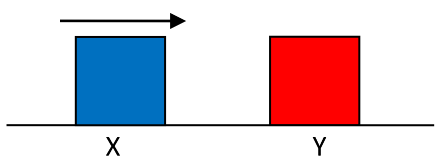
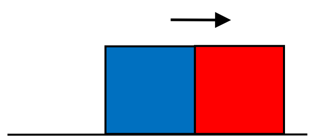
Teacher notes
Topic A

Maximum possible energy transfer

Block X of mass *m* approaches a stationary block Y of mass *M* on a smooth surface with velocity v_X. The bodies stick together after the collision.





Why is the kinetic energy lost the maximum possible?

Imagine that we look at things from the point of view of an observer who moves with the same velocity as the center of mass of the two blocks. This velocity is

$$v_{CM} = \frac{mv_X}{m+M}$$

In this reference frame (the CM frame) X has velocity (we use capital letters for quantities in the CM frame)

$\underset{CM}{V_X} = \underset{lab}{v_X} - v_{CM} = v_X - \frac{mv_X}{m+M} = \frac{Mv_X}{m+M}$ and Y has velocity $\underset{CM}{V_Y} = \underset{lab}{0} - v_{CM} = -\frac{mv_X}{m+M}$.

In the CM frame X has momentum $P_X = mV_X = \frac{Mmv_X}{m+M}$ and Y has momentum $P_Y = -MV_Y = -\frac{Mmv_X}{m+M}$. I.e. $P_Y = -P_X$ and in this frame the total momentum is 0. So the total momentum will be zero after the collision as well. Let us call $P_X = P$.

The total kinetic energy before the collision in the CM frame is then

$$K = \frac{P^2}{2m} + \frac{P^2}{2M} = \frac{m+M}{2mM}P^2$$

After the collision X and Y have equal and opposite momenta, Q and $-Q$ (because the total momentum is zero) so the new kinetic energy is

$$K' = \frac{Q^2}{2m} + \frac{Q^2}{2M} = \frac{m+M}{2mM}Q^2$$

The loss in kinetic energy is then

$$\Delta K = \frac{m+M}{2mM}P^2 - \frac{m+M}{2mM}Q^2 = \frac{m+M}{2mM}(P^2 - Q^2)$$

This is as large as possible when $Q = 0$.

This is the momentum of each block in the CM frame after the collision. So, the velocity of each block in the CM frame is also zero. This means that in the lab frame the velocities of each block after the collision v'_X and v'_Y are given by:

X $\quad \underset{\text{CM}}{0} = \underset{\text{lab}}{v'_X} - v_{CM} \Rightarrow v'_X = v_{CM} = \dfrac{mv_X}{m+M}$

Y $\quad \underset{\text{CM}}{0} = \underset{\text{lab}}{v'_Y} - v_{CM} \Rightarrow v'_Y = v_{CM} = \dfrac{mv_X}{m+M}$

This means the blocks move together. Indeed, if the blocks stick together their common velocity would be found from

$$mv_X + 0 = (m+M)u \Rightarrow u = \frac{mv_X}{m+M}$$

in complete agreement with the analysis above.

It is not at all difficult to extend this analysis to the case where both blocks move initially.

In outline:

$$v_{CM} = \frac{mv_X + Mv_Y}{m+M}$$

$$V_X = v_X - v_{CM} = v_X - \frac{mv_X + Mv_Y}{m+M} = \frac{M(v_X - v_Y)}{m+M}$$

$$V_Y = v_Y - v_{CM} = v_Y - \frac{mv_X + Mv_Y}{m+M} = \frac{-m(v_X - v_Y)}{m+M}$$

$$P_X = mV_X = \frac{mM}{m+M}(v_X - v_Y)$$

$$P_Y = MV_Y = -\frac{Mm}{m+M}(v_X - v_Y)$$

$$P_Y = -P_X = -P$$

$$K = \frac{P^2}{2m} + \frac{P^2}{2M} = \frac{m+M}{2mM}P^2$$

$$K' = \frac{Q^2}{2m} + \frac{Q^2}{2M} = \frac{m+M}{2mM}Q^2$$

$$\Delta K = \frac{m+M}{2mM}(P^2 - Q^2)$$

With Q = 0 again,

The velocities of each block after the collision v'_X and v'_Y are given by:

X $\quad 0 = v'_X - v_{CM} \Rightarrow v'_X = v_{CM} = \dfrac{mv_X + Mv_Y}{m+M}$

Y $\quad 0 = v'_Y - v_{CM} \Rightarrow v'_Y = v_{CM} = \dfrac{mv_X + Mv_Y}{m+M}$

The velocity of the two blocks is the same-they move together.

A diversion

If the collision is elastic, $\Delta K = 0$ and so $P^2 = Q^2$ implying $P = \pm Q$. If $P = +Q$ the velocities of each block after the collision, v'_X and v'_Y, are given by:

X $\quad +\dfrac{Mmv_X}{m+M} = m(v'_X - v_{CM}) \Rightarrow v'_X = +\dfrac{Mv_X}{m+M} + \dfrac{mv_X}{m+M} = v_X$

Y $\quad -\dfrac{Mmv_X}{m+M} = M(v'_Y - v_{CM}) \Rightarrow v'_Y = -\dfrac{mv_X}{m+M} + \dfrac{mv_X}{m+M} = 0$

This is not a physical solution though: it is as if X went through Y without anything happening. This is not possible in mechanics (but it does happen in wave motion-pulses go through each other!).

If, on the other hand, $P = -Q$:

X $\quad -\dfrac{Mmv_X}{m+M} = m(v'_X - v_{CM}) \Rightarrow v'_X = -\dfrac{Mv_X}{m+M} + \dfrac{mv_X}{m+M} = -\dfrac{(M-m)v_X}{m+M}$

Y $\quad \dfrac{Mmv_X}{m+M} = M(v'_Y - v_{CM}) \Rightarrow v'_Y = \dfrac{mv_X}{m+M} + \dfrac{mv_X}{m+M} = \dfrac{2mv_X}{m+M}$

This is what we would deduce if we applied momentum and energy conservation:

$mv_X + 0 = mv'_X + Mv'_Y$ and $\frac{1}{2}mv_X^2 + 0 = \frac{1}{2}mv'^2_X + \frac{1}{2}Mv'^2_Y$

and then solved for v'_X and v'_Y .